Solution of Integral Equations By Laplace Transform

Prepared By

Najem A. MOHAMMAD

M.Sc. mathematics /Faculty of Science and Health / mathematics Department

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Abstract

We will discuss the modern form of operation calculus based on the Laplace transformation and investigate its application to the solution of linear volterra integral equations (V.I.E) .as we can see , the Laplace transformation is especially effective to solve volterra integral equations of a first and second kind. Also we used the Laplace transformation to find the solution of (V.I.E.1) and (V.I.E.2) in order to express these solutions, tested using several examples.

1.1 introduction

The theory and application of integral equations is an important subject in applied mathematics. Integral equations are used as mathematical models for many and varied physical situation and integral equation also occur as reformulation of other mathematical problems [2].

There are several methods to finding the solution of integral equation. Laplace transformation is one of them, which is used to solve ordinary difference equations, and integral equations [4, 5].

1.2 Classification of linear integral equation :-[3,4,5,6]

Definition (1.1):- An equation which contains on unknown function under integral symbol is called integral equation in [illegible]([illegible]) and can be expressed as:-

[illegible] * [illegible] = [illegible] + [illegible] ∫ [illegible] **(1.1)**

Where [illegible]([illegible]), [illegible]([illegible]) are known continuous function on [[illegible], [illegible]] of [illegible], [illegible]([illegible]) is unknown function and [illegible]([illegible], [illegible]) is called kernel of the integral equation, [illegible] is called the scalar parameter.

Definition (1.2):-the integral equation (1.1) is called the integral equation of first kind when [illegible] = [illegible] which is

[illegible] = [illegible] ∫ [illegible] 1.3

Definition (1.3):-the integral equation (1.1) is called the integral equation of second kind when [illegible] which is

$$[illegible] \qquad (1.2)$$

Definition (1.4):-if **(a)** and ([illegible]) are constant real numbers the equation (1.2) and (1.3) are called Fredholm integral equations of the first and second kind respectively

$$[illegible] \qquad (1.4)$$

$$[illegible] \qquad (1.5)$$

Definition (1.5):-the integral equations (1.2) and (1.3) are called volterra integral equation of the first and second kind when [illegible] and [illegible] is constant hence the integral equations becomes

$$[illegible] \qquad (1.6)$$

$$[illegible] \qquad (1.7)$$

Definition (1.6):-When the kernel $K(x,t)$ depends only on the $(x-t)$, it is termed the difference kernel $K(x,t) = K(x-t)$ then $K(x,t)$ is called deference kernel.

1.3 Laplace transformation:-[1, 3,7]

Definition (1.6):- Let $f(x)$ be real valued function defined on $x > 0$ for any real number **(p)**, and

$L\{f(x)\} = F(p) = \int_0^{\infty} e^{-px} f(x)\,dx$, is finite converges, then this function **f(x)** is called the Laplace transform.

1.4 Laplace transformation of some functions:-

1) $L\{1\} = \frac{1}{p}$
2) $L\{e^{ax}\} = \frac{1}{p-a}$
3) $L\{x^{n}\} = \frac{n!}{p^{n+1}}$
4) $L\{\sin(ax)\} = \frac{a}{p^{2}+a^{2}}$
5) $L\{\sinh(ax)\} = \frac{a}{p^{2}-a^{2}}$

2.1:-Laplace transform for solving ([illegible]) : we have here two sections, in section **(2.2)** using Laplace transform for solving ([illegible]) with defference kernal and example are solvid as indicate in eaxmple ,(2.1)and (2.2) in section (2.3) we reduce voltteral integar equation of the first kind to voltteral integral equation of the second kind and solve it by using the aim of Laplace transform and inverse of Laplace transform example are solved. Which are showed in examples (2.3) and (2.4) .

[illegible]: − [2,3].

Let the function [illegible] be original function, and let

[illegible] = [illegible] and [illegible] = [illegible]

Then [illegible] [illegible] = [illegible] **(1)**

The integral on the right of (1) is called the convolution of the function [illegible] and [illegible] and denoted by the symbol [illegible] ∗ [illegible]. Thus a product of transform is also a transform namely the transform of the convolution of the original functions then [illegible] [illegible] = [illegible] ∗ [illegible].

[illegible] — [2,3].

When the Volterra integral equation of the first kind has a difference

[illegible] = [illegible] — [illegible] as follows

λ [illegible] **(2.1)**

We applied Laplace transforms of both sides of equation (2.1) with convolution theorem we get

[illegible] = λ [illegible] $\frac{1}{\lambda}$ **(2.2)**

Then we get [illegible] = [illegible] **(2.3)**

By taking inverse of Laplace transformation of equation (2.1) the solution equation is

[illegible] = $\frac{1}{\lambda}$ [illegible] **(2.4)**

[illegible] — Solve the volterra integral equation of a first kind

$$\sin(x) = \lambda \int_0^x e^{x-t} u(t) dt$$

Solution:-

By using Laplace transformation of both sides of equation

[illegible] = [illegible] and [illegible] = [illegible]

By equation (2.2) we get

$$\frac{[illegible]}{[illegible] + [illegible]} = \lambda \frac{[illegible]}{[illegible] - [illegible]} [illegible]([illegible])$$

$$[illegible] = \frac{1}{\lambda} \frac{\frac{[illegible]}{[illegible] + [illegible]}}{\frac{[illegible]}{[illegible] - [illegible]}} = \frac{1}{\lambda} \frac{[illegible] - [illegible]}{[illegible] + [illegible]} = \frac{1}{\lambda} \left(\frac{[illegible]}{[illegible] + [illegible]} - \frac{[illegible]}{[illegible] + [illegible]} \right)$$

Take the inverse [illegible] of both side of equation, we get

$$[illegible]^{-1} [illegible] = \frac{1}{\lambda} [illegible]^{-1} \left(\frac{[illegible]}{[illegible] + [illegible]} - \frac{[illegible]}{[illegible] + [illegible]} \right)$$

$[illegible] = \frac{1}{\lambda} ([illegible] - [illegible]([illegible]))$ is exact solution

2.4 Reduce volterra integral equation of the first kind to volterra integral equation of the second kind[[illegible]].

The volterra equation of the first kind [illegible] $= \lambda$ [illegible] (2.5)

Can by reduce to volterra equation of the second kind when [illegible] $\neq 0$ since if we diferention both sides of equation (2.5) with respect to [illegible] using (generalized formula [3]) on the integral we get

[illegible] $=$ [illegible] $+ \lambda$ [illegible] (2,6)

This can easily be re written

$$u(x) = \frac{1}{\lambda k(x,x)} \frac{dg}{dx} - \int_0^x \frac{1}{k(x,x)} \frac{\partial k(x,t)}{\partial x} u(t)dt \qquad (2.7)$$

As a volterra integral equation of the second kind with the non-homogenous term

[illegible] $=$ [illegible] and the kernal

[illegible] $=$ [illegible]

Thus [illegible] $=$ [illegible] $+$ [illegible] (2.8)

Equation (2.8) is volterra integral equation of second kind and can-be solved by adding Laplace transformation and inverse of Laplace transformation .

2.5 Using Laplace transform to solve volterra integral equation of the second kind with convolution theorem [2,3].

When the kernel $\square(\square,\square)$ depends only on the difference $\square - \square$, it is termed the difference kernel

$\square(\square,\square) = \square(\square - \square)$. The volterra equation of the second kind with difference kernel $\square(\square - \square)$

$$\square(\square) = \square(\square) + \square \int \square(\square - \square)\,\square(\square)\,\square\square \qquad (2.9)$$

Now has an integral in the form of the Laplace convolution product

$$\int_{\square}^{\square} \square(\square - \square)\,\square(\square)\,\square\square = \square \ast \square$$

Hence the voltera equation with difference kernel (2.9) taking the Laplace transform (2.9) and by assuming $\square(\square)$, $\square(\square)$ and $\square(\square)$ be the Laplace transforms of $\square(\square)$, $\square(\square)$ and $\square(\square)$ respectively realize form the convolution theorem that

$$\square(\square) = \square(\square) + \square\square(\square)\square(\square) \qquad (2.10)$$

$$\square(\square) = \frac{\square(\square)}{\square - \square\square(\square)}, \quad \square\square(\square) \neq \square \qquad (2.11)$$

Taking the inverse Laplace transform of (2.11), we get

$$\square(\square) = \square^{-1}\left[\frac{\square(\square)}{\square - \square\square(\square)}\right], \quad \square\square(\square) \neq \square \qquad (2.12)$$

which can be evaluated with the add of Laplace transform pairs.

Example (2.3):-

Solve the following volterra equations of the first kind after reducing it to a volterra equation of the second kind and use the Laplace transform to solve resulting integral equation

[illegible]

Solution:-

[illegible]

[illegible]

then [illegible] $\neq$ [illegible]

By using equation (2.7) we get

[illegible]

[illegible]

[illegible] is V.I.E of 2^{nd}.K

taking Laplace transformation of both side

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

taking inverse Laplace transformation

[illegible]

[illegible] is exact solution.

Thank you 4 listening